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New varieties of Gowdy spacetimes

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ABSTRACT

Gowdy spacetimes are generalized to admit two commuting spatial *local* Killing vectors, and some new varieties of them are presented, which are all closely related to Thurston's geometries. Explicit spatial compactifications, as well as the boundary conditions for the metrics are given in a systematic way. A short comment on an implication to their dynamics toward the initial singularity is made.

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I. INTRODUCTION

Spatially compact inhomogeneous spacetimes admitting two commuting spatial Killing vectors are known as Gowdy spacetimes [1], which are recently paid large attention as favorable models for the studies of the asymptotic behavior toward the cosmological initial singularity [2–6]. Since Gowdy spacetimes provide the simplest inhomogeneous cosmologies, it seems natural to use them in such a kind of studies first developed based on Bianchi homogeneous cosmologies [7] by Belinski et.al. [8]. This article’s aim is not, however, to elucidate the dynamics of Gowdy spacetimes, but to point out some new features pertaining to their varieties, which may open new windows to look at the initial singularity.

As is well known, Bianchi homogeneous cosmologies are classified by the local structures of simply transitive three dimensional groups, Bianchi I~IX groups. Spatially compact Bianchi cosmologies [9–11] are more diverse than the open ones, since possible compact topologies are in general very diverse for each Bianchi (universal covering) cosmology. Nevertheless, Gowdy spacetimes having lower symmetry than these homogeneous ones are known to have very limited varieties, i.e., only T^3 , $S^2 \times S^1$, S^3 , and lens spaces as their possible spatial topologies. Why is their diversity so poor? In fact, all the Bianchi cosmologies except types VIII and IX admit two commuting Killing vectors, and moreover, compactifications are possible except for types IV and VI_{a ≠ 0} [12,9]. Since these compact Bianchi models can be thought of as, if exist, homogeneous limits of compact inhomogeneous models which admit two commuting Killing vectors, it seems that Gowdy models should be more diverse. The solution to this paradox is in whether the Killing vectors are local or global. That is, the restriction for the possible topologies of Gowdy spacetimes is a consequence of the definition that the two commuting Killing vectors must be *globally* defined, while spatially compact Bianchi cosmologies admit in general only *local* Killing vectors.

If we consider Killing vectors for the simplification of Einstein’s equation, which is local in nature, the imposition of the globality of the Killing vectors is evidently unnecessary. We therefore generalize Gowdy spacetimes in this article to admit two commuting *local* Killing vectors. (This type of generalization was also considered by Rendall [13] in a different approach from ours.) We will actually find that there exist rich (topologically infinitely many) varieties of Gowdy spacetimes.

As is well known in theory of three dimensional topology, Thurston [14] enumerated eight types of homogeneous 3-manifolds, called *model geometries*, S^3 , E^3 , H^3 , $S^2 \times E^1$, $H^2 \times E^1$, $\widetilde{SL}(2, \mathbf{R})$, nilgeometry(Nil), and solvegeometry(Sol), and proved in essence that any compact three dimensional manifold which admit a locally homogeneous Riemannian metric is a compact quotient of one of these eight types of homogeneous manifolds. (See Sec.3.8 of Ref. [14].) We effectively utilize these model geometries, as in the compact locally homogeneous cases [9,10].

In the next section, we show how we can apply Thurston's theorem to find (generalized) Gowdy spacetimes, and in the subsequent two sections present three new types of Gowdy spacetimes as examples. The final section is devoted to conclusions, including a comment on the dynamics of the three Gowdy spacetimes.

II. POSSIBLE TOPOLOGIES

To find possible topologies of Gowdy spacetimes, we consider a ‘homogenization’ of a compact Riemannian 3-manifold (M, h_{ab}) which admit two commuting local Killing vectors. Namely, suppose we can smoothly deform the metric h_{ab} preserving the two local Killing vectors and can take a locally homogeneous limit (M, h_{ab}^{lh}) of the Gowdy space (M, h_{ab}) . Since the universal cover $(\tilde{M}, \tilde{h}_{ab}^{\text{lh}})$ of (M, h_{ab}^{lh}) is homogeneous, it must be one of the BKS_N types, i.e., Bianchi homogeneous 3-manifolds and the Kantowski-Sachs-Nariai (KSN) homogeneous 3-manifold [15,16]. On the other hand, since the homogenization (M, h_{ab}^{lh}) is compact, (M, h_{ab}^{lh}) must be homeomorphic to a compact quotient of one of the eight Thurston's model geometries.

A (model) geometry is the pair (\tilde{M}, G) of a manifold \tilde{M} and a group G of diffeomorphisms on \tilde{M} , such that G acts transitively on \tilde{M} with compact isotropy subgroups. Since G acts transitively on \tilde{M} , (\tilde{M}, G) can be thought of as an equivalence class of the homogeneous manifolds whose isometry group is isomorphic to (or includes a subgroup isomorphic to) G [10]. We for convenience call (\tilde{M}, G) a *subgeometry* of (\tilde{M}, G') if G is a subgroup of other transitive group G' with compact isotropy subgroups. Moreover, if geometry (\tilde{M}, G) is not a subgeometry of any geometry, then we call (\tilde{M}, G) a *maximal* geometry, and if (\tilde{M}, G) does not have any subgeometry, then we call (\tilde{M}, G) a *minimal* geometry. While Thurston's eight geometries are maximal geometries, all the BKS_N types except Bianchi IV and VI_{a ≠ 0} can be thought of as the minimal geometries of Thurston's geometries [9,12]. This correspondence helps apply Thurston's geometries to spacetime models.

The group G of the geometry (\tilde{M}, G) which corresponds to $(\tilde{M}, \tilde{h}_{ab}^{\text{lh}})$ should include a subgroup isomorphic to one of \mathbf{R}^2 , $\mathbf{R} \times U(1)$, and $U(1) \times U(1)$, corresponding to the commuting two Killing vectors. All Thurston's geometries satisfy this condition. More precisely, what include \mathbf{R}^2 are geometries E^3 , H^3 , $H^2 \times E^1$, $\widetilde{SL}(2, \mathbf{R})$, Nil, and Sol. Geometry $S^2 \times E^1$ includes $\mathbf{R} \times U(1)$, and S^3 does $U(1) \times U(1)$. Moreover, all the minimal geometries of these Thurston's geometries except Bianchi VIII ($\widetilde{SL}(2, \mathbf{R})$ as the maximal one) and IX (S^3 , similarly) also satisfy this condition. Any compact quotient for all the homogeneous 3-manifolds except Bianchi VIII and IX can therefore be a homogenization of a Gowdy space. Even for Bianchi VIII and IX, they can be a homogenization if imposing a fourth Killing vector on them.

This observation tells what homogeneous manifold can be the universal cover of a homogeneous limit of a Gowdy space. Once fixed such a homogenization with isometry group G , the universal cover of the corresponding Gowdy space can be determined as follows. That is, we may find the subgroup H of G such that the actions are smooth along the two commuting Killing vectors, but discrete along the third one. The H -invariant metric is the universal cover metric of the Gowdy space (or spacetime), if the fundamental group can be represented in H . In fact, the H -invariance is a necessary condition for the universal cover to admit a (spatially) compact quotient, so that we need to check that. As we will see, it is easy to write such an H -invariant metric if using the invariant 1-forms for the corresponding Bianchi type. The KSN type has already been discussed in Ref. [1], so we do not consider it in this article.

In the following, we show how these ideas work by presenting three examples, which are all new.

III. GOWDY ON NIL $\times\mathbf{R}$

As a first example let us consider Nil, which possesses Bianchi II as its minimal geometry. Note that the Bianchi II homogeneous spaces are characterized by the following commutation relations for the three Killing vectors

$$[\xi_1, \xi_2] = -\xi_3, \quad [\xi_2, \xi_3] = 0, \quad [\xi_3, \xi_1] = 0. \quad (1)$$

In terms of coordinate basis they can be represented by

$$\xi_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \quad \xi_2 = \frac{\partial}{\partial y}, \quad \xi_3 = \frac{\partial}{\partial z}. \quad (2)$$

For future use, we take this opportunity to write the finite actions generated by these ξ_i 's

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a + x \\ b + y \\ c + z + ay \end{pmatrix}, \quad (3)$$

where the component vectors

$$\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \equiv e^{c\xi_3} e^{b\xi_2} e^{a\xi_1} \mid a, b, c \in \mathbf{R} \right\} \quad (4)$$

form Bianchi II group G_{II} . ($e^{a\xi_i}$ denotes the one-parameter group of diffeomorphisms generated by ξ_i .) Note that the “spatial point” (x, y, z) is the image of the origin $(0, 0, 0)$ by

the “group element” (x, y, z) , so that Eq.(3) itself gives the multiplication rule in Bianchi II group.

From the commutation relations (1), we find two possible choices of obtaining a Gowdy spacetime model, i.e., one (Type 1) is to keep ξ_2 and ξ_3 as the two commuting Killing vectors and consider inhomogeneity along ξ_1 , and the other (Type 2) is to keep ξ_1 and ξ_3 as Killing vectors and consider inhomogeneity along ξ_2 . We consider the Type 1 first.

Note that the invariant 1-forms of Bianchi II, given by

$$\sigma^1 = dx, \quad \sigma^2 = dy, \quad \sigma^3 = dz - xdy \quad (5)$$

are globally defined on $\tilde{M} = \mathbf{R}^3$, so that we can expand any metric on \tilde{M} by these 1-forms

$$dl^2 = h_{ij}\sigma^i\sigma^j, \quad (6)$$

with globally defined metric functions h_{ij} . Note also that the subgroup H_{II} of G_{II} of which action is discrete along ξ_1 is formed by

$$\left\{ \begin{pmatrix} 2m\pi \\ b \\ c \end{pmatrix} \mid m \in \mathbf{Z}, b, c \in \mathbf{R} \right\}. \quad (7)$$

(The choice of the interval for the first component is arbitrary. Our choice of 2π is just for definiteness.) Since the invariant 1-forms (5) are invariant under H_{II} (and G_{II} by definition), the metric (6) is invariant under H_{II} iff so are the metric functions h_{ij} . This requirement is equivalent to that h_{ij} ’s depend only on x and are periodic with period 2π , i.e.,

$$h_{ij} = h_{ij}(x) = h_{ij}(x + 2\pi). \quad (8)$$

The homogeneous limit can be achieved when $h_{ij} = \text{constants}$. So, we have found the inhomogeneous metric of (the universal cover of) a Gowdy space, given by Eq.(6) with the boundary condition (8).

Now, we can write down the “appropriate” spacetime metric by [1] imposing the two-surface orthogonality and choosing the isothermal coordinates for the reference surface. After all, we obtain the spacetime metric

$$ds^2 = e^{-\lambda/2}t^{-1/2}(-dt^2 + (\sigma^1)^2) + R[e^P(\sigma^2)^2 + 2e^PQ\sigma^2\sigma^3 + (e^PQ^2 + e^{-P})(\sigma^3)^2], \quad (9)$$

where P, Q, R and λ are functions of t and x and are periodic in x with period 2π . We have followed the parameterization of a recent paper [6] to make comparisons easier. The isometry group for the spacetime is unchanged, given by H_{II} (7).

This metric should be considered as the universal cover metric \tilde{g}_{ab} of a Gowdy spacetime $(M \times \mathbf{R}, g_{ab})$. Any topology of M is possible if the fundamental group $\pi_1(M)$ can be represented in H_{II} . One can choose such a $\pi_1(M)$ in the list of compact quotients modeled on Nil, presented in Ref. [9]. For example, choose the manifold “ $b/1(n)$ ”, characterized by

$$\pi_1(M_n) = \langle g_1, g_2, g_3; [g_1, g_2] = g_3^n, [g_1, g_3] = 1, [g_2, g_3] = 1 \rangle, \quad (10)$$

where n is a positive integer parameterizing the family $b/1$ of topologies on Nil. Then using the multiplication rule (3), we can find the representation up to conjugations

$$\begin{aligned} \Gamma_n &= \{g_1, g_2, g_3\} \\ &= \left\{ \begin{pmatrix} 2p\pi \\ g_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2q\pi \\ g_2^2 \\ g_2^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{2\pi}{n}(pg_2^2 - qg_1^2) \end{pmatrix} \right\}, \end{aligned} \quad (11)$$

where $p, q \in \mathbf{Z}$, and $g_1^2, g_2^2, g_2^3 \in \mathbf{R}$. Thus, we have obtained the Gowdy spacetime $(M_n \times \mathbf{R}, g_{ab}) = (\tilde{M}_n \times \mathbf{R}, \tilde{g}_{ab})/\Gamma_n$, where the action of Γ_n is defined by Eq.(3). We may think of the real parameters (g_1^2, g_2^2, g_2^3) as the moduli parameters for the spacetime.

The Type 2 can be obtained by interchanging the roles of σ^1 and σ^2 . For example, the metric can be obtained from Eq.(9) by transforming $\sigma^1 \rightarrow \sigma^2$, $\sigma^2 \rightarrow \sigma^1$, $P(t, x) \rightarrow P(t, y)$, etc.. The isometry group thereof is formed by the actions

$$\left\{ \begin{pmatrix} a \\ 2m\pi \\ c \end{pmatrix} \mid m \in \mathbf{Z}, a, c \in \mathbf{R} \right\}. \quad (12)$$

As can be easily checked, we can represent the fundamental group (10) into this isometry group in a form similar to Eq.(11).

Let us consider the vacuum Einstein equations. To be specific, we consider the Type 1. (As for the final results, we will present for both Types 1 and 2.) Note that if neglecting the boundary conditions, our metric (9) is essentially the same as the metric given in Ref. [1]

$$ds^2 = e^{-\lambda/2}t^{-1/2}(-dt^2 + dx^2) + R[e^P dy^2 + 2e^P Q dy dz + (e^P Q^2 + e^{-P}) dz^2], \quad (13)$$

where P , Q , R , and λ are functions of t and x . In fact, our metric (9) is obtained from this one by transforming

$$P \rightarrow P + \ln[(1 - xQ)^2 + x^2 e^{-2P}], \quad Q \rightarrow \frac{Q(1 - xQ)^2 - xe^{-2P}}{(1 - xQ)^2 + x^2 e^{-2P}} \quad (14)$$

(, though this does not preserve the periodicity of P and Q). Moreover, the role of R as the area function of the group orbits, consisting of flat T^2 's [17], is the same for both metrics.

This can be checked by noticing that the natural volume element of the second term in the metric (9) is given by $R dy \wedge dz$. As a result, the function R of our metric satisfies the same key equation as that of metric (13), i.e., [1]

$$\partial_{tt}R - \partial_{xx}R = 0. \quad (15)$$

This equation can also be checked by a direct substitution into the vacuum Einstein equation. Then, since the group orbits for our spatial manifold do not degenerate everywhere (except at the initial singularity), R can be taken as $t(\equiv e^{-\tau})$, as in the Gowdy model on $T^3 \times \mathbf{R}$ [1]. $t = 0$ ($\tau = +\infty$) corresponds to the initial singularity. With this choice of R , the remaining independent Einstein's equations for our metric (9) are found by a direct calculation to be

$$\begin{aligned} \ddot{P} - e^{-2\tau} P'' - e^{2P} \dot{Q}^2 + e^{-2\tau} [e^{2P} (Q^2 \pm Q')^2 \pm 2Q' - e^{-2P}] &= 0, \\ \ddot{Q} - e^{-2\tau} Q'' + 2\dot{P}\dot{Q} - 2e^{-2\tau} [P'Q' \pm (P' \mp Q)(Q^2 + e^{-2P})] &= 0, \end{aligned} \quad (16)$$

and

$$\begin{aligned} \lambda' - 2(P' \mp 2Q)\dot{P} \mp 2[e^{2P}(Q^2 \pm Q') - 1]\dot{Q} &= 0, \\ \dot{\lambda} - \dot{P}^2 - e^{2P}\dot{Q}^2 - e^{-2\tau} [e^{2P}(Q^2 \pm Q')^2 + P'^2 + 2Q^2 \mp 2(Q' + 2P'Q) + e^{-2P}] &= 0, \end{aligned} \quad (17)$$

where dot and dash denote τ and x (or y for the Type 2) derivatives, respectively. The upper and lower signs are for the Type 1 and Type 2, respectively. Note that P and Q are not constrained, since λ does not appear in Eqs.(16). This is one of the advantages of Gowdy models. The integrability condition for the constraint equations (17) for λ is automatically satisfied with Eqs.(16). The boundary conditions for P , Q , and λ are the spatially periodic ones. The Hamiltonian for the dynamical equations (16) can be guessed from straightforwardly reducing the Einstein-Hilbert action with the metric (9). It is given by

$$H = \frac{1}{2} \int_0^{2\pi} d\theta [\pi_P^2 + e^{-2P} \pi_Q^2] + e^{-2\tau} [(P' \mp 2Q)^2 - 2(Q^2 \pm Q') + e^{2P} (Q^2 \pm Q')^2 + e^{-2P}], \quad (18)$$

where π_P and π_Q are the conjugate momenta of P and Q , respectively. The integration measure θ is x for the Type 1, or y for the Type 2.

IV. GOWDY ON SOL×R AND VII₀ × R

Next examples correspond to Bianchi VI₀ and VII₀ as minimal geometries. Their maximal geometries are Sol and E^3 , respectively. Since the local structures for Bianchi VI₀ and

VII_0 are similar, we treat them in parallel in this section. The basic procedure is the same as that of the previous section.

We first observe the commutation relations of the sets of the three Killing vectors of the Bianchi groups

$$[\xi_1, \xi_2] = 0, [\xi_2, \xi_3] = -\xi_1, [\xi_3, \xi_1] = \xi_2 : \text{VI}_0, \quad (19)$$

$$[\xi_1, \xi_2] = 0, [\xi_2, \xi_3] = -\xi_1, [\xi_3, \xi_1] = -\xi_2 : \text{VII}_0. \quad (20)$$

(We write “tags” to distinguish the two types as above.) In terms of coordinate basis,

$$\xi_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \xi_2 = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \xi_3 = \frac{\partial}{\partial z} - x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} : \text{VI}_0, \quad (21)$$

$$\xi_1 = \frac{\partial}{\partial x}, \xi_2 = \frac{\partial}{\partial y}, \xi_3 = \frac{\partial}{\partial z} - y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} : \text{VII}_0. \quad (22)$$

The Bianchi groups are formed by the finite actions of ξ_i 's, which we take as

$$G_{\text{VI}_0} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \equiv e^{\frac{a}{2}(\xi_1+\xi_2)} e^{\frac{b}{2}(\xi_1-\xi_2)} e^{c\xi_3} \mid a, b, c \in \mathbf{R} \right\}, \quad (23)$$

$$G_{\text{VII}_0} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \equiv e^{a\xi_1} e^{b\xi_2} e^{c\xi_3} \mid a, b, c \in \mathbf{R} \right\}. \quad (24)$$

The actions or multiplications are given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a + e^{-c}x \\ b + e^c y \\ c + z \end{pmatrix} : \text{VI}_0, \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \mathcal{R}_c \begin{pmatrix} x \\ y \end{pmatrix} \\ c + z \end{pmatrix} : \text{VII}_0, \quad (25)$$

where \mathcal{R}_c is the rotation matrix by angle c .

From the commutation relations (19) and (20), we find sole possibility of obtaining the Gowdy spaces, the ones inhomogeneous along ξ_3 . Using the invariant 1-forms

$$\sigma^1 = \frac{1}{\sqrt{2}}(e^z dx + e^{-z} dy), \sigma^2 = \frac{1}{\sqrt{2}}(-e^z dx + e^{-z} dy), \sigma^3 = dz : \text{VI}_0, \quad (26)$$

$$\sigma^1 = \cos z dx + \sin z dy, \sigma^2 = -\sin z dx + \cos z dy, \sigma^3 = dz : \text{VII}_0, \quad (27)$$

and the periodic functions

$$\begin{aligned} P &= P(\tau, z) = P(\tau, z + z_0), Q = Q(\tau, z) = Q(\tau, z + z_0), \\ \lambda &= \lambda(\tau, z) = \lambda(\tau, z + z_0), \end{aligned} \quad (28)$$

we can immediately write the same spacetime metric for the two types as

$$ds^2 = e^{-\lambda/2}e^{\tau/2}(-e^{-2\tau}d\tau^2 + (\sigma^3)^2) + e^{-\tau}[e^P(\sigma^1)^2 + 2e^PQ\sigma^1\sigma^2 + (e^PQ^2 + e^{-P})(\sigma^2)^2]. \quad (29)$$

As we will see later, the period z_0 is specified depending upon the topology. The isometry groups H_{VI_0} and H_{VII_0} for the spacetime metric (29) is formed by the discrete actions along ξ_3 and can be written in the same form

$$\left\{ \begin{pmatrix} a \\ b \\ mz_0 \end{pmatrix} \mid a, b \in \mathbf{R}, m \in \mathbf{Z} \right\}. \quad (30)$$

In the following, we consider the vacuum Einstein equations first, since the compactifications must be discussed separately for each universal cover.

In the metric (29), we wrote the area function of the group orbit as $e^{-\tau} = t$, since, as in the Nil case, the group orbits do not degenerate except at the initial singularity for both types. The remaining Einstein equations are then found to be

$$\begin{aligned} \ddot{P} - e^{-2\tau}P'' - e^{2P}\dot{Q}^2 + e^{-2\tau}[e^{2P}(Q' + Q^2 \mp 1)^2 + 2Q' - e^{-2P}] &= 0, \\ \ddot{Q} - e^{-2\tau}Q'' + 2\dot{P}\dot{Q} - 2e^{-2\tau}[P'Q' + (P' - Q)(e^{-2P} + Q^2 \mp 1)] &= 0, \end{aligned} \quad (31)$$

and

$$\begin{aligned} \lambda' - 2(P' - 2Q)\dot{P} - 2[e^{2P}(Q' + Q^2 \mp 1) - 1]\dot{Q} &= 0, \\ \dot{\lambda} - \dot{P}^2 - e^{2P}\dot{Q}^2 - e^{-2\tau}[(P' - 2Q)^2 - 2(Q' + Q^2 \mp 1) + e^{2P}(Q' + Q^2 \mp 1)^2 + e^{-2P}] &= 0, \end{aligned} \quad (32)$$

where dot and dash denote, respectively, τ and z derivatives. The upper and lower signs are for Sol(VI₀) and VII₀, respectively. The integrability condition for the constraint equations (32) for λ is automatically satisfied with Eqs.(31). The Hamiltonian for the dynamical equations (31) is given by

$$H = \frac{1}{2} \int_0^{z_0} dz [\pi_P^2 + e^{-2P}\pi_Q^2] + e^{-2\tau} \left[(P' - 2Q)^2 - 2(Q' + Q^2) + e^{2P}(Q' + Q^2 \mp 1)^2 + e^{-2P} \right]. \quad (33)$$

Compactification for Gowdy on Sol×R: We can take any topology of M of the Gowdy spacetime $(M \times \mathbf{R}, g_{ab})$ on Sol×R if we can represent the fundamental group $\pi_1(M)$ into H_{VI_0} (30) together with, if needed, the disconnected components defined with the discrete isometry

$$h : (x, y, z) \rightarrow (-x, -y, z). \quad (34)$$

As an explicit example, we consider the major sequence of compact quotients presented in Ref. [9]. The fundamental groups are parameterized by an integer n such that $|n| > 2$, and given by

$$\pi_1(M_n) = \langle g_1, g_2, g_3; [g_1, g_2] = 1, g_3 g_1 g_3^{-1} = g_2, g_3 g_2 g_3^{-1} = g_1^{-1} g_2^n \rangle. \quad (35)$$

(This does not, however, exhaust all the compact quotients [18].) We find that the representations are, up to conjugations, given by

$$\Gamma_n = \left\{ \begin{pmatrix} \alpha u_1 \\ \beta u_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha v_1 \\ \beta v_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_0 \end{pmatrix}, \right\} \quad (36)$$

for $n > 2$, and

$$\Gamma_n = \left\{ \begin{pmatrix} \alpha u_1 \\ \beta u_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha v_1 \\ \beta v_2 \\ 0 \end{pmatrix}, h \circ \begin{pmatrix} 0 \\ 0 \\ z_0 \end{pmatrix}, \right\} \quad (37)$$

for $n < -2$, with $\alpha, \beta \in \mathbf{R}$. In these representations, (u_1, v_1) , (u_2, v_2) , and z_0 are determined in such a way that $\text{sign}(n)e^{-z_0}$ and $\text{sign}(n)e^{z_0}$ are the eigenvalues of matrix $\begin{pmatrix} 0 & 1 \\ -1 & n \end{pmatrix}$, and the corresponding normalized eigenvectors are (u_1, v_1) and (u_2, v_2) , respectively. In particular, $e^{z_0} = |n + \sqrt{n^2 - 4}|/2$. We thus have obtained the Gowdy spacetime $(M_n \times \mathbf{R}, g_{ab}) = (\tilde{M}_n \times \mathbf{R}, \tilde{g}_{ab})/\Gamma_n$, where the universal cover metric \tilde{g}_{ab} is given by Eq.(29) with 1-forms (26).

Compactification for Gowdy on VII₀ × R: The Gowdy model which is most frequently picked up in the literature is the one on $T^3 \times \mathbf{R}$, which would have been obtained by our procedure starting from Bianchi I. However, another $T^3 \times \mathbf{R}$ model can be obtained from Bianchi VII₀, which we pick up here.

The fundamental group of T^3 is the infinite group with three commuting generators;

$$\pi_1(T^3) = \langle g_1, g_2, g_3; [g_1, g_2] = 1, [g_2, g_3] = 1, [g_3, g_1] = 1 \rangle. \quad (38)$$

The general solution of the representation into “ G_{VII_0} ” has already given in Eq.(97) of Ref. [9];

$$\Gamma = \left\{ \begin{pmatrix} g_1^1 \\ g_1^2 \\ 2l\pi \end{pmatrix}, \begin{pmatrix} g_2^1 \\ g_2^2 \\ 2m\pi \end{pmatrix}, \begin{pmatrix} g_3^1 \\ g_3^2 \\ 2n\pi \end{pmatrix} \right\}, \quad (39)$$

where $g_1^1, g_1^2, g_2^1, g_2^2, g_3^1, g_3^2 \in \mathbf{R}$, and $l, m, n \in \mathbf{Z}$. We immediately notice that this representation is effective even in “ H_{VII_0} ” if we choose the period z_0 to 2π . So, this representation with $z_0 = 2\pi$ is the general solution. The only difference from the locally homogeneous case is that there are no effective conjugations. The representation (39) therefore gives the final form of the covering group. Finally, we have obtained the Gowdy spacetime $(T^3 \times \mathbf{R}, g_{ab}) = (\mathbf{R}^4, \tilde{g}_{ab})/\Gamma$, where the universal cover metric \tilde{g}_{ab} is given by Eq.(29) with 1-forms (27).

V. CONCLUSIONS

We have generalized Gowdy spacetimes to admit two commuting *local* Killing vectors. By this generalization, we gained rich varieties of new Gowdy spacetimes, but any advantages the original has, e.g., the simplicity of the vacuum Einstein equations, have not been lost. In this sense, the original definition demanding global existence of the two commuting Killing vectors was too much restricted. Our generalization is natural and useful for physical applications.

We have presented three new Gowdy spacetimes, on $\text{Nil} \times \mathbf{R}$, on $\text{Sol} \times \mathbf{R}$, and on $\text{VII}_0 \times \mathbf{R}$, which are closely related to Thurston's geometries, Nil, Sol, and E^3 , respectively. We have given not only the vacuum Einstein equations with boundary conditions but also an explicit representation of the covering group for each case. These three new models and the one called $T^3 \times \mathbf{R}$ model in the literature have common features like (1) the group orbits do not degenerate everywhere except at the initial singularity, and (2) there are two dynamical variables (i.e., P and Q). These features make the four models very similar. In fact, the only essential difference is the boundary conditions for the metric. Since we wrote the spacetime metrics in a suitable way for each case, their Einstein equations look different each other, but the boundary conditions for the metric functions are the same, simply periodic. We could have wrote the spacetime metric in a common form, e.g., like Eq.(13), but in that case, the boundary conditions for the metric functions would have taken inconvenient forms, as we have seen in the case of Nil.

It is worth noting that we can interpret the difference of the boundary conditions for the metrics as the difference of the “background metrics”. For example, the spatial metric of the (conventional) $T^3 \times \mathbf{R}$ model can be smoothly deformed locally flat, so that we can think that the background is flat, while, say, the spatial metric of the $\text{Nil} \times \mathbf{R}$ model cannot be deformed locally flat but locally Bianchi II, so that the background is the Bianchi II locally homogeneous curved space in this case.

Here, we comment on the dynamics near the initial singularity of our three models. The Gowdy spacetime on $T^3 \times \mathbf{R}$ is conjectured [3] to be asymptotically velocity term dominated (AVTD) [2]. Recent investigation by Berger and Garfinkle [6] supports this, *except for* measure-zero nongeneric spatial points. They succeeded to explain the phenomenon by a potential picture, and showed the nongeneric points correspond to the points where $Q' = 0$. In our new models, such points correspond to points such that $Q' \pm Q^2 = 0$ (for the Types 1 and 2 of Nil) and $Q' + Q^2 \mp 1 = 0$ (for Sol and VII_0). Note that the points where $Q' = 0$ is inevitable, since Q is periodic, but our four conditions are not necessarily satisfied in any spatial point. One may therefore expect that the Gowdy $\text{Nil} \times \mathbf{R}$, $\text{Sol} \times \mathbf{R}$, and $\text{VII}_0 \times \mathbf{R}$ models are AVTD *everywhere*, so the “curved backgrounds” improve the AVTD behavior.

As a final remark, an extension of our method to the so-called $U(1)$ models [19] and other

similar ones would also be possible, which is now under development, as well as a complete classification and further study of Gowdy models.

Note after the completion of this work: Recently, Weaver, Isenberg, and Berger [20] applied a Gowdy model on $\text{Sol} \times \mathbf{R}$ with magnetic field to examine the Mixmaster behavior toward the initial singularity. The boundary conditions imposed there is consistent with ours, though they look different, since they used a metric similar to Eq.(13).

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- [1] R. H. Gowdy, Phys. Rev. Lett. **27**, 826 (1971); 1102 (E); Ann. Phys. **83**, 203 (1974).
- [2] J. Isenberg and V. Moncrief, Ann. Phys. **199**, 84 (1990).
- [3] B. Grubišić and V. Moncrief, Phys. Rev. D **47**, 2371 (1993).
- [4] B. K. Berger and V. Moncrief, Phys. Rev. D **48**, 4676 (1993).
- [5] S. D. Hern and J. M. Stewart, preprint, DAMTP R-97/41 (gr-qc/9708038, 1997).
- [6] B. K. Berger and D. Garfinkle, preprint (gr-qc/9710102, 1997).
- [7] See, e.g., L. D. Landau, E. M. Lifshitz, Classical theory of Fields (MIT, Reading, 1971).
- [8] V. A. Belinski, I. M. Khalatnikov and E. M. Lifshitz, Adv. in Phys. **19**, 525 (1970); *ibid* **31**, 639 (1982).
- [9] T. Koike, M. Tanimoto and A. Hosoya, J. Math. Phys. **35**, 4855 (1994).
- [10] M. Tanimoto, T. Koike and A. Hosoya, J. Math. Phys. **38**, 350 (1997).
- [11] M. Tanimoto, T. Koike and A. Hosoya, J. Math. Phys. **38**, 6560 (1997).
- [12] H. V. Fagundes, Phys. Rev. Lett. **54**, 1200 (1985).
- [13] A. D. Rendall, Commun. Math. Phys. **189**, 145 (1997).

- [14] W. P. Thurston, Three-Dimensional Geometry and Topology Vol.1 (Princeton University Press, Princeton, 1997).
- [15] H. Nariai, Sci. Rep. Tohoku Univ., I, **34**, 160 (1950); *ibid.*, **35**, 62 (1951).
- [16] R. Kantowski and R. K. Sachs, J. Math. Phys. **7**, 443 (1967).
- [17] The group actions are well defined on $(\tilde{M}, \tilde{h}_{ab})$, and the orbits form Euclid planes E^2 . The compactification of $(\tilde{M}, \tilde{h}_{ab})$ compactifies these E^2 to two-tori T^2 , which we call the group orbits as well, though the group actions are defined only locally on the quotient.
- [18] H. Kodama, Prog. Theor. Phys. **99**, 173 (1998).
- [19] V. Moncrief, Ann. Phys. **167**, 118 (1986).
- [20] M. Weaver, J. Isenberg, and B.K. Berger, preprint, (gr-qc/9712055, 1997).